Localising Groups

Riley Moriss

December 5, 2025

Given a finitely generated abelian group, structure theory tells us that it is of the form

$$\mathbb{Z}^n \bigoplus_\alpha \mathbb{Z}/p_\alpha^{k_\alpha}\mathbb{Z}, \quad k_\alpha \in \mathbb{N}, \quad p_\alpha \text{ prime}$$

The question is how can we get at the "local" peices without perhaps knowing the whole group.

1 Lemmas

Lemma ([AM18], Prop 2.14)**.** *For $\mathbb{Z}$ modules M_α, N tensored over $\mathbb{Z}$*

$$\left(\bigoplus_\alpha M_\alpha\right) \otimes N \cong \bigoplus_\alpha (M_\alpha \otimes N)$$

Lemma (Proof on Stackexchange)**.**

$$\mathbb{Z}/m\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$$

Lemma.

$$\mathbb{Z}/m\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} \cong 0$$

Proof. (Following Ex.2 in Tensor product chapter [AM18]) Consider the exact sequence of $\mathbb{Z}$ modules

$$0 \to (n) \to \mathbb{Z} \to \mathbb{Z}/(n) \to 0$$

We tensor it with $\mathbb{Q}$ to get an exact sequence ($\mathbb{Q}$ is flat by the 0AUW lemma below)

$$0 \to (n) \otimes \mathbb{Q} \to \mathbb{Q} \to (\mathbb{Z}/(n)) \otimes \mathbb{Q} \to 0$$

by exactness and the first isomorphism theorem we know that

$$(\mathbb{Z}/(n)) \otimes \mathbb{Q} \cong \mathbb{Q}/\big((n) \otimes \mathbb{Q}\big)$$

but because $(n) \cong \mathbb{Z}$ as a $\mathbb{Z}$ module (note that to see that if $A \cong B$ then $A \otimes M \cong B \otimes M$ we can apply the tensor product to the isomorphism between A and B, at least when M is flat as $\mathbb{Q}$ is then it will preserve isomorphisms)

$$(n) \otimes \mathbb{Q} \cong \mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q}$$

we have that $(\mathbb{Z}/(n)) \otimes \mathbb{Q} \cong \mathbb{Q}/\mathbb{Q} = 0$.

Recall that $\mathbb{Z}_{(p)}$ is $\mathbb{Z}$ localised at the ideal (p), this inverts everything outside of the ideal. Strictly we define the multiplicative set $S = \mathbb{Z} - (p)$ and then define $\mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$.

Lemma.

$$\mathbb{Z}_{(p)} \otimes \mathbb{Z}/q^n\mathbb{Z} \cong \begin{cases} \mathbb{Z}/q^n\mathbb{Z}, & q = p \\ 0, & q \neq p \end{cases}$$

Proof. [AM18, Prop 3.5] states that $\mathbb{Z}_{(p)} \otimes M \cong M_{(p)}$. So we need to calculate only $\mathbb{Z}/q^n\mathbb{Z}$ localised at (p). When $p = q$ then we are inverting elements that are not in (p), that is elements of $\mathbb{Z}$ coprime to p, all of which are already inverible in $\mathbb{Z}/q^n\mathbb{Z}$, so this has no effect. When $p \neq q$ then $q \in \mathbb{Z} - (p)$, that is it is not in the ideal (p), hence it is one of the elements that we are inverting. But so is q^n. Therefore we have to have a group such that the image of q^n is both the multiplicative and additive unit, this can only be in the trivial group.

Lemma (Stacks Project, Lemma 0AUW)**.** $\mathbb{Z}_{(p)}$ *is flat as a* $\mathbb{Z}$ *module.*

Proof. Just to be extra clear, $\mathbb{Z}_{(p)}$ is a $\mathbb{Q}$ subring and therefore torsion free.

2 Discussion

So now if we have our finitely generated abelian group as above then because the tensor product distributes over the sum when we tensor product it with a group we will get a direct sum back and it will be of the peices tensored.

If we want to find the rank of the free part we tensor with $\mathbb{Q}$ as this will send all the torsion parts to the trivial group and give you a $\mathbb{Q}$ vector space of the same rank as the rank of the $\mathbb{Z}$ in the original group.

If we want to find the torsion peices it is clear that we cannot just tensor with the cyclic groups (or the finite fields) as this merely "counts" the number of peices that have a common factor for instance

$$\mathbb{Z}/4\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$$

On the other hand we can see that tensoring with the localised integers will give us the required peice. Thus we have basically proven that for a finitely generated abelian group G then

$$G \cong \mathbb{Z}^{\mathrm{rank}_{\mathbb{Q}} G \otimes \mathbb{Q}} \bigoplus_{p \text{ prime}} G \otimes \mathbb{Z}_{(p)}$$

we call $G \otimes \mathbb{Z}_{(p)}$ the p-primary part of G.

References

[AM18] Michael Francis Atiyah and Ian G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley series in mathematics. CRC Press, Taylor & Francis Group, Boca Raton London New York, 2018.